

# Path-Integral bosonization of a non-local interaction and its application to the study of 1-d many-body systems

C.M.Naón<sup>a,b</sup>, M.C.von Reichenbach<sup>a,b</sup> and M.L.Trobo<sup>a,b</sup>

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## Abstract

We extend the path-integral approach to bosonization to the case in which the fermionic interaction is non-local. In particular we obtain a completely bosonized version of a Thirring-like model with currents coupled by general (symmetric) bilocal potentials. The model contains the Tomonaga-Luttinger model as a special case; exploiting this fact we study the basic properties of the 1-d spinless fermionic gas: fermionic correlators, the spectrum of collective modes, etc. Finally we discuss the generalization of our procedure to the non-Abelian case, thus providing a new tool to be used in the study of 1-d many-body systems with spin-flipping interactions.

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<sup>a</sup> Depto. de Física. Universidad Nacional de La Plata. CC 67, 1900 La Plata, Argentina.

<sup>b</sup> Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina.

# 1 Introduction

Two dimensional models are a fruitful testing ground for new ideas and methods in QFT. In particular the Thirring model [1] [2] and its non-Abelian version [3] have been extensively explored to shed light on such important phenomena as confinement and asymptotic freedom. Moreover, the connection between the Thirring model and Statistical Mechanics systems [4] [5] has made their range of interest even wider. In particular, in Solid State Physics, the Tomonaga-Luttinger model (TL) [6] [7] [8], which describes a one-dimensional gas of highly correlated spinless particles, can be understood in terms of a Thirring-like system. Recently, striking developments in the field of nanofabrication have allowed to build one-dimensional semiconductors [9], thus adding a more practical motivation for the theoretical study of this type of low-dimensional models [10].

From the Field Theory point of view, the Thirring and Gross-Neveu models were successfully analyzed in the path-integral framework, by means of a decoupling change in the path-integral variables [11], [12]. This method provided a functional integral version [13] of Coleman's Abelian bosonization [14] and prefigured its non-Abelian extension [15]. Later on it was also applied to the Kondo problem [16] and to the computation of critical exponents in 2D Ising and Baxter models [17].

In this work we study a non-local Thirring-like (NLT) model with Euclidean action given by:

$$S = \int d^2x \bar{\Psi} i \not{\partial} \Psi - \frac{g^2}{2} \int d^2x d^2y [V_{(0)}(x, y) J_0(x) J_0(y) + V_{(1)}(x, y) J_1(x) J_1(y)] \quad (1.1)$$

where  $J_\mu = \bar{\Psi} \gamma_\mu \Psi$  and  $V_{(\mu)}(x, y)$  is an arbitrary function of two variables. Note that for  $V_{(0)} = V_{(1)} = \delta^2(x - y)$  one recovers the usual covariant Thirring model.

Our motivation is twofold. On the one hand we want to extend the path-integral approach to bosonization, developed in [11], to the case in which a non-local interaction contributes to the total action. On the other hand, we also wish to show how our techniques can be applied to the study of 1-d many-body systems. In Section 2 we describe the method and obtain the non-local bosonized action. In Section 3 we compute the 2-point fermionic

correlator. Later on, in Section 4 we observe that for  $V_{(1)} = 0$  and  $V_{(0)} = \delta(x_0 - y_0)v(x_1 - y_1)$ , the action (1.1) describes a many-body system, similar to the TL model. This allows us to obtain known properties of a Luttinger liquid [18],[19], in a straightforward way. In Section 5 we generalize our procedure to the non-Abelian case. For this model, which contains the spin-1/2 TL system as a special case, we obtain a bosonized action, including a Wess-Zumino term. If one disregards spin-flipping processes, our method yields the spectrum of charge and spin-density modes. Finally, in Section 6 we gather our results and conclusions.

## 2 The Model and the Method

We start by considering the Euclidean vacuum functional

$$Z = N \int D\bar{\Psi} D\Psi e^{-S} \quad (2.1)$$

where  $N$  is a normalization constant and  $S$  is given by (1.1). As it is habitual in the path-integral approach to the usual Thirring model, one eliminates this quartic fermionic interaction by introducing an auxiliary vector field  $A_\mu$ . As we shall see, in the present non-local case one needs one more auxiliary field to achieve the same goal. In order to depict this procedure we first observe that  $S$  can be splitted in the form

$$S = S_0 + S_{int} \quad (2.2)$$

where

$$S_0 = \int d^2x \bar{\Psi} i \not{\partial} \Psi, \quad (2.3)$$

and

$$S_{int} = -\frac{g^2}{2} \int d^2x J_\mu K_\mu. \quad (2.4)$$

In this last expression  $J_\mu$  is the usual fermionic current,

$$J_\mu(x) = \bar{\Psi}(x) \gamma_\mu \Psi(x), \quad (2.5)$$

and  $K_\mu$  is a new current defined as

$$K_\mu(x) = \int d^2y V_{(\mu)}(x, y) J_\mu(y). \quad (2.6)$$

Please note that no sum over repeated indices is implied when a subindex  $(\mu)$  is involved. The partition function can now be written as

$$Z = N \int D\bar{\Psi} D\Psi D\tilde{A}_\mu D\tilde{B}_\mu \exp[-\{S_0 + \int d^2x [\tilde{A}_\mu \tilde{B}_\mu - \frac{g}{\sqrt{2}}(\tilde{A}_\mu J_\mu + \tilde{B}_\mu K_\mu)]\}] \quad (2.7)$$

where we have used the following representation of the functional delta:

$$\delta(C_\mu) = \int D\tilde{A}_\mu e^{-\int d^2x \tilde{A}_\mu C_\mu}. \quad (2.8)$$

Using now equations (2.5) and (2.6), the fermionic piece of the action can be cast in the form

$$S_0 - \frac{g}{\sqrt{2}} \int d^2x (\tilde{A}_\mu J_\mu + \tilde{B}_\mu K_\mu) = \int d^2x \bar{\Psi} [i\not{\partial} - \frac{g}{\sqrt{2}} \gamma_\mu (\tilde{A}_\mu + \tilde{B}_\mu)] \Psi, \quad (2.9)$$

where we have defined

$$\bar{B}_\mu(x) = \int d^2y V_{(\mu)}(y, x) \tilde{B}_\mu(y). \quad (2.10)$$

For later convenience we shall invert (2.10) in the form

$$\tilde{B}_\mu(x) = \int d^2y b_{(\mu)}(y, x) \bar{B}_\mu(y), \quad (2.11)$$

with  $b_{(\mu)}(y, x)$  satisfying

$$\int d^2y b_{(\mu)}(y, x) V_{(\mu)}(z, y) = \delta^2(x - z). \quad (2.12)$$

Equation (2.9) suggests the following change of auxiliary variables:

$$\frac{1}{\sqrt{2}}(\tilde{A}_\mu + \tilde{B}_\mu) = A_\mu, \quad (2.13)$$

$$\frac{1}{\sqrt{2}}(\tilde{A}_\mu - \tilde{B}_\mu) = B_\mu. \quad (2.14)$$

Employing equation (2.11), the purely bosonic piece of the action reads

$$\begin{aligned}
\int d^2x \tilde{A}_\mu(x) \tilde{B}_\mu(x) &= \frac{1}{2} \int d^2x d^2y \{ b_{(0)}(y, x) A_0(x) A_0(y) + \\
&+ b_{(1)}(y, x) A_1(x) A_1(y) - b_{(0)}(y, x) B_0(x) B_0(y) + \\
&+ b_{(1)}(y, x) B_1(x) B_1(y) + \\
&- [b_{(0)}(y, x) - b_{(0)}(x, y)] A_0(x) B_0(y) + \\
&- [b_{(1)}(y, x) - b_{(1)}(x, y)] A_1(x) B_1(y) \}. \tag{2.15}
\end{aligned}$$

From now on we shall restrict our study to the case in which the bilocal functions  $V_{(\mu)}$  and  $b_{(\mu)}$  are symmetric so that the last two terms in the integrand of (2.15) vanish. Under these conditions the partition function of the system is given by

$$Z = N_1 \int DA_\mu DB_\mu \det(i\cancel{D} + g\cancel{A}) e^{-S[A, B]}, \tag{2.16}$$

where  $S[A, B]$  coincides with the r.h.s. of (2.15) for  $b_{(0)}(y, x) = b_{(0)}(x, y)$  and  $b_{(1)}(y, x) = b_{(1)}(x, y)$ . Note that the Jacobian associated with the change  $(\tilde{A}, \tilde{B}) \rightarrow (A, B)$ , is field-independent and can be absorbed in the normalization constant ( $N \rightarrow N_1$ ).

We have been able to express  $Z$  in terms of a fermionic determinant. This fact will enable us to apply the machinery of the path-integral approach to bosonization, first developed in the context of local theories, to the present non-local case. But before we do this some remarks are in order. First of all it is worthwhile noting that, as a consequence of the change of bosonic variables (eqs.(2.13) and (2.14)), the effect of the non-local interaction has been completely transferred to the purely bosonic piece of the action,  $S[A, B]$ . On the other hand we see that the  $B$ -field is completely decoupled from both the  $A$ -field and the fermion field. Keeping this in mind, it is instructive to try to recover the partition function corresponding to the usual covariant Thirring model ( $b_{(0)}(y, x) = b_{(1)}(x, y) = \delta^2(x - y)$ ), starting from (2.16). In doing so one readily discovers that  $B_\mu$  describes a negative-metric state whose contribution must be factorized and absorbed in  $N_1$  in order to get a sensible answer for  $Z$ . This procedure parallels, in the path-integral framework, the operator approach of Klaiber [2], which precludes the use of an indefinite-metric Hilbert space. If one follows the same prescription in the present non-local case, the result is:

$$Z = N_2 \int DA_\mu \det(i\cancel{D} + g\cancel{A}) e^{-S[A]}, \quad (2.17)$$

where  $N_2$  includes the contribution of the "non-local ghost"  $B_\mu$  and  $S[A]$  is the  $A$ -dependent part of (2.15) in the symmetric case.

At this stage we perform a chiral change in the fermionic path-integral measure:

$$\Psi(x) = e^{-g[\gamma_5 \Phi(x) + i\eta(x)]} \chi(x) \quad (2.18)$$

$$\bar{\Psi}(x) = \bar{\chi}(x) e^{-g[\gamma_5 \Phi(x) - i\eta(x)]} \quad (2.19)$$

$$D\bar{\Psi}D\Psi = J_F[\Phi, \eta] D\bar{\chi}D\chi, \quad (2.20)$$

where  $\Phi$  and  $\eta$  are scalar fields and  $J_F[\Phi, \eta]$  is the Fujikawa Jacobian [20] whose non-triviality is due to the non-invariance of the path-integral measure under chiral transformations. In 1 + 1 dimensions the auxiliary field  $A_\mu$  can be always decomposed into longitudinal and transverse parts in the form

$$A_\mu(x) = \epsilon_{\mu\nu} \partial_\nu \Phi(x) + \partial_\mu \eta(x). \quad (2.21)$$

Using now (2.20) and (2.21) one obtains

$$\det(i\cancel{D} + g\cancel{A}) = J_F[\Phi, \eta] \det i\cancel{D}. \quad (2.22)$$

Concerning the fermionic Jacobian, it can be computed following Fujikawa's procedure . Since its detailed evaluation has been given several times in the literature [11], here we just quote the final result:

$$\ln J_F[\Phi, \eta] = \frac{g^2}{2\pi} \int d^2x \Phi \square \Phi - S_{reg} \quad (2.23)$$

As usual,  $J_F$  contributes with a kinetic term for  $\Phi(x)$ . We have also written an additional term  $S_{reg}$  in (2.23) in order to emphasize that the computation of  $J_F$  requires a regularization prescription. For local gauge theories with Dirac fermions, such as QED<sub>2</sub> and QCD<sub>2</sub>, one is naturally led to consider a regularization scheme that preserves gauge invariance ( $S_{reg} = 0$  in this case). When the vector field does not correspond to a gauge field, one can choose a more general regulator (the usual Thirring model [1] and the chiral Schwinger

model [21] are examples in which regularization ambiguities take place). Evidently, the same occurs in the present case. Following the prescriptions of Ref.[22] for the evaluation of  $S_{reg}$ , we obtain

$$S_{reg} = -\frac{\alpha}{2\pi} \int d^2x \Phi \square \Phi \quad (2.24)$$

where  $\alpha$  is an arbitrary parameter which can be determined on gauge invariance grounds.

Having eliminated the negative-metric fields, using now eq.(2.21), eq.(2.15) contributes with kinetic-like terms for  $\Phi$  and  $\eta$  fields (they become kinetic terms in the local case, when the potentials  $b_\mu$  are delta functions). Derivative couplings between  $\Phi$  and  $\eta$  are also present as an effect of non-covariance ( $b_{(0)} \neq b_{(1)}$ ).

Taking into account that

$$DA \equiv DA_0 DA_1 = J_{bos} D\Phi D\eta, \quad (2.25)$$

where  $J_{bos}$  is a trivial Jacobian (in the sense that it does not depend on the fields), and putting together the above described contributions to (2.17), we finally get

$$Z = N \int D\Phi D\eta e^{-S_{eff}} \quad (2.26)$$

where  $N = N_2 J_{bos} \det i\cancel{D}$  and

$$\begin{aligned} S_{eff} = & \frac{g^2 + \alpha}{2\pi} \int d^2x (\partial_\mu \Phi)^2 + \\ & + \frac{1}{2} \int d^2x d^2y [b_{(0)}(y, x) \partial_1 \Phi(x) \partial_1 \Phi(y) + b_{(1)}(y, x) \partial_0 \Phi(x) \partial_0 \Phi(y)] + \\ & + \frac{1}{2} \int d^2x d^2y [b_{(0)}(y, x) \partial_0 \eta(x) \partial_0 \eta(y) + b_{(1)}(y, x) \partial_1 \eta(x) \partial_1 \eta(y)] + \\ & + \int d^2x d^2y [b_{(0)}(y, x) \partial_0 \eta(x) \partial_1 \Phi(y) - b_{(1)}(y, x) \partial_1 \eta(x) \partial_0 \Phi(y)] \quad (2.27) \end{aligned}$$

Equations (2.26) and (2.27) constitute the main result of this Section. Thus, we have been able to extend the path-integral approach to bosonization, previously applied to the solution of local QFT's, to a Thirring-like model of fermions with a non-local interaction term. More specifically, we have shown

the equivalence between the fermionic partition function (2.1) and the functional integral (2.26) corresponding to the two bosonic degrees of freedom  $\Phi$  and  $\eta$  with dynamics governed by (2.27). The contribution to this action coming from the fermionic Jacobian (the first term in the r.h.s of (2.27)) exactly coincides with the one which is obtained in the local case [12]. On the other hand, the effect of non-locality is contained in the remaining terms, through the inverse potentials  $b_\mu(x, y)$ . Note that, even in the non-local case,  $\Phi$  and  $\eta$  become decoupled for  $b_{(0)} = b_{(1)}$ . Of course, when  $b_{(0)} = b_{(1)} = \delta^2(x - y)$ , one recovers the bosonic version of the local Thirring model.

The spectrum of this bosonic model can be more easily analyzed in momentum space. Indeed, by Fourier transforming (2.27) one obtains

$$S_{eff} = \frac{1}{(2\pi)^2} \int d^2p \{ \hat{\Phi}(p) \hat{\Phi}(-p) A(p) + \hat{\eta}(p) \hat{\eta}(-p) B(p) + \hat{\Phi}(p) \hat{\eta}(-p) C(p) \}, \quad (2.28)$$

where

$$A(p) = \frac{g^2 + \alpha}{2\pi} p^2 + \frac{1}{2} [\hat{b}_{(0)}(p) p_1^2 + \hat{b}_{(1)}(p) p_0^2], \quad (2.29)$$

$$B(p) = \frac{1}{2} [\hat{b}_{(0)}(p) p_0^2 + \hat{b}_{(1)}(p) p_1^2], \quad (2.30)$$

$$C(p) = [\hat{b}_{(0)}(p) - \hat{b}_{(1)}(p)] p_0 p_1, \quad (2.31)$$

and  $\hat{\Phi}$ ,  $\hat{\eta}$  and  $\hat{b}_{(\mu)}$  are the Fourier transforms of  $\Phi$ ,  $\eta$  and  $b_{(\mu)}$  respectively.

Eq. (2.28) can be easily diagonalized through the change

$$\begin{aligned} \hat{\Phi} &= \hat{\zeta} - \frac{C}{2A} \hat{\xi} \\ \hat{\eta} &= \hat{\xi}. \end{aligned} \quad (2.32)$$

We then have the following propagators for  $\hat{\zeta}$  and  $\hat{\xi}$ :



$$G_{\zeta}^{-1}(p) = \lambda p^2 + \frac{1}{2}[\hat{b}_{(0)}p_1^2 + \hat{b}_{(1)}p_0^2] \quad (2.33)$$

$$G_{\xi}^{-1}(p) = \frac{\lambda p^2[\hat{b}_{(0)}p_0^2 + \hat{b}_{(1)}p_1^2] + \frac{\hat{b}_{(0)}\hat{b}_{(1)}}{2}p^4}{2\lambda p^2 + \hat{b}_{(0)}p_1^2 + \hat{b}_{(1)}p_0^2}, \quad (2.34)$$

where  $\lambda = \frac{g^2 + \alpha}{2\pi}$  and  $\hat{b}_{(0)}, \hat{b}_{(1)}$  are functions of  $p$ . These expressions are further simplified in the case  $\hat{b}_{(0)} = \hat{b}_{(1)}$ . In particular, when  $\hat{b}_{(0)} = \hat{b}_{(1)} \propto \frac{1}{p^2}$  the  $\hat{\zeta}$  field acquires a mass, whereas  $\hat{\xi}$  becomes a non propagating field.

### 3 Two-point fermionic correlations

Let us now use the results of the precedent section to compute the fermionic propagator

$$\langle \Psi(x) \bar{\Psi}(y) \rangle = \begin{pmatrix} 0 & G_+(x, y) \\ G_-(x, y) & 0 \end{pmatrix}. \quad (3.1)$$

Once we have performed the decoupling change of variables given by eqs. (2.18) and (2.19), the non-vanishing components of the Green function are factorized in the form

$$G_{\pm}(x, y) = G_{\pm}^{(0)}(x, y) B_{\pm}(x, y), \quad (3.2)$$

where

$$G_{\pm}^{(0)}(z) = \frac{1}{2\pi |z|^2} (z_0 \pm iz_1) \quad (3.3)$$

and

$$B_{\pm}(x, y) = \langle e^{\pm g[\Phi(y) - \Phi(x)]} e^{\pm ig[\eta(y) - \eta(x)]} \rangle_{eff}. \quad (3.4)$$

In this expression  $\langle \quad \rangle_{eff}$  is a v.e.v. to be computed with the action given by eq.(2.27). Working in momentum-space this bosonic factor can be written as

$$B_{\pm}(x, y) = \frac{\int D\hat{\Phi}D\hat{\eta} e^{-[S_{eff}+S_{\pm}(x,y)]}}{\int D\hat{\Phi}D\hat{\eta} e^{-S_{eff}}} \quad (3.5)$$

with  $S_{eff}$  given by eq.(2.28) and

$$S_{\pm}(x, y) = -\frac{g}{(2\pi)^2} \int d^2p [\pm\hat{\Phi}(p) \pm i\hat{\eta}(p)]D(p; x, y) \quad (3.6)$$

where

$$D(p; x, y) = e^{-ip \cdot x} - e^{-ip \cdot y}. \quad (3.7)$$

Now  $B_{\pm}$  can be easily evaluated by performing the change

$$\begin{aligned} \hat{\Phi}(p) &= \hat{\varphi}(p) + E_{\pm}(p; x, y) \\ \hat{\eta}(p) &= \hat{\rho}(p) + F_{\pm}(p; x, y) \end{aligned} \quad (3.8)$$

where  $\hat{\varphi}$  and  $\hat{\rho}$  are the new quantum variables and  $E_{\pm}(p)$  and  $F_{\pm}(p)$  are classical functions chosen in the form

$$E_{\pm}(p) = \pm g[iC(p) - 2B(p)] \frac{D(p; x, y)}{\Delta(p)} \quad (3.9)$$

$$F_{\pm}(p) = \pm g[C(p) - 2iA(p)] \frac{D(p; x, y)}{\Delta(p)} \quad (3.10)$$

with  $A, B$  and  $C$  defined in (2.29-2.31) and

$$\Delta(p) = C^2(p) - 4A(p)B(p) \quad (3.11)$$

Putting all this together, the result is

$$B_{\pm}(x, y) = \exp\left\{-\frac{g^2}{\pi} \int d^2p \sin^2\left[\frac{p \cdot (x - y)}{2}\right] \frac{B(p) - A(p) \mp iC(p)}{\Delta(p)}\right\} \quad (3.12)$$

Of course, in order to go further in this computation one needs to specify the functions  $\hat{b}_{(\mu)}(p)$  which determine the integrand in (3.12). The simplest case at hand is the one corresponding to the usual Thirring model, which

can be used to check the consistency of our more general calculation. Setting  $\hat{b}_{(0)} = \hat{b}_{(1)} = 1$ , we get

$$B_{\pm}^{Thirring}(z) = \exp\left\{\frac{1}{2\pi} \frac{\left(\frac{g^2}{\pi}\right)^2}{\left(1 + \frac{g^2}{\pi}\right)} \int \frac{d^2p}{p^2} \sin^2 \frac{p \cdot z}{2}\right\} \quad (3.13)$$

The integral in the above expression can be easily performed just by taking care of the ultraviolet divergence. We then obtain

$$B_{\pm}^{Thirring}(z) = c |z|^{-\frac{1}{2}(g^2\pi)^2/(1+g^2\pi)} \quad (3.14)$$

where  $c$  is a constant which depends on the ultraviolet cutoff. This formula displays the exact continuously varying exponent typical of the covariant Thirring model (see, for instance, Ref. [12]).

## 4 Connection with the Tomonaga-Luttinger model

We shall apply in this Section the approach developed in previous Sections to the TL model [6] [7] [8]. This model describes a non-relativistic gas of spinless and massless particles (electrons) in which the dispersion relation is taken to be linear. The free-particle Hamiltonian is given by

$$H_0 = v_F \int dx \Psi^\dagger(x) (\sigma_3 p - p_F) \Psi(x) \quad (4.1)$$

where  $v_F$  and  $p_F$  are the Fermi velocity and momentum respectively ( $v_F p_F$  is a convenient origin for the energy scale).  $\sigma_3$  is a Pauli matrix and  $\Psi$  is a column bispinor with components  $\Psi_1$  and  $\Psi_2$  ( $\Psi^\dagger = (\Psi_1^\dagger \ \Psi_2^\dagger)$ ). The function  $\Psi_1(x)$  [ $\Psi_2(x)$ ] is associated with the motion of particles in the positive [negative]  $x$  direction. The interaction piece of the Hamiltonian, when only forward scattering is considered, is

$$H_{int} = \int dx \int dy \sum_{a,b} \Psi_a^\dagger(x) \Psi_a(x) V_{ab}(x, y) \Psi_b^\dagger(y) \Psi_b(y) \quad (4.2)$$

where  $a, b = 1, 2$ , and the interaction matrix is parametrized in the form

$$V_{ab} = \begin{pmatrix} v_1 & v_2 \\ v_2 & v_1 \end{pmatrix}. \quad (4.3)$$

Using the imaginary-time formalism one can show that the finite-temperature [23] [24] action for this problem becomes

$$\begin{aligned} S_{TL} &= \int_0^\beta d\tau \int dx \{ \bar{\Psi} \gamma_0 (\partial_\tau - v_p p_F) \Psi + v_F \bar{\Psi} \gamma_1 \partial_x \Psi \} \\ &+ \int_0^\beta d\tau \int dx \int dy \sum_{a,b} \Psi_a^\dagger \Psi_a(x, \tau) V_{ab}(x, y) \Psi_b^\dagger \Psi_b(y, \tau). \end{aligned} \quad (4.4)$$

For simplicity, in this Section we shall set  $v_F = 1$  and consider the case  $v_1 = v_2$  in (4.3) [8]. We shall also restrict ourselves to the zero temperature limit ( $\beta \rightarrow \infty$ ). Under these conditions it is easy to verify that  $S_{TL}$  coincides with the non-local Thirring model discussed in the precedent Section, provided that the following identities hold:

$$\begin{aligned} g^2 &= 2 \\ V_{(0)}(x, y) &= v_1(x, y) = v_2(x, y) = v(x_1 - y_1) \delta(x_0 - y_0) \\ V_{(1)} &= 0 \end{aligned} \quad (4.5)$$

Of course one has also to make the shift  $\bar{\Psi} \gamma_0 \partial_0 \Psi \rightarrow \bar{\Psi} \gamma_0 (\partial_0 - p_F) \Psi$  and identify  $x_0 = \tau$ ,  $x_1 = x$ .

One then can employ the method described in Section 2 in order to study the Tomonaga-Luttinger model. This model has been previously studied, through a different functional approach, by D.K. Lee and Y. Chen [25]. These authors, however, avoided the use of the decoupling technique applied here. As we shall see, our approach will be particularly useful when considering spin-flipping interactions, i.e. the non-Abelian extension of the model (see Section 5).

Let us first focus our attention to the dispersion relations corresponding to the elementary excitations of the model at hand. These states correspond to the normal modes whose dynamics is governed by the action (2.28). As it is well-known, the spectrum of these modes is obtained from the poles

of the corresponding propagators. Alternatively, one can write the effective Lagrangian as

$$L_{eff} = \frac{1}{(2\pi)^2} \begin{pmatrix} \hat{\Phi} & \hat{\eta} \end{pmatrix} \begin{pmatrix} A & C/2 \\ C/2 & B \end{pmatrix} \begin{pmatrix} \hat{\Phi} \\ \hat{\eta} \end{pmatrix} \quad (4.6)$$

(with A, B and C defined in (2.29)-(2.31)) and solve the equation

$$\Delta(p) = 0, \quad (4.7)$$

with  $\Delta(p)$  defined in (3.11). Going back to real frequencies :  $p_0 = i\omega$ ,  $p_1 = q$ , (4.7) yields a biquadratic equation for  $\omega$ . The relevant solution is

$$\omega_-^2(q) = \frac{\hat{b}_{(1)}}{\hat{b}_{(0)}} \frac{2\lambda + \hat{b}_{(0)}}{2\lambda + \hat{b}_{(1)}} q^2. \quad (4.8)$$

There is also a free dispersion relation which appears due to the fact that the density fields are related to  $\Psi$  and  $\eta$  fields through first derivatives. This point will be discussed in detail in the next Section (see the paragraph following eq.(5.25)).

Inserting now the identities (4.5) in (4.8) and setting  $\alpha = 0$  ( $\lambda = \frac{q^2}{2\pi}$ ) we obtain

$$\omega_-^2(q) = q^2 \left\{ 1 + \frac{2v(q)}{\pi} \right\} \quad (4.9)$$

which is the well-known result for the spectrum of the charge-density excitations of the TL model in the Mattis-Lieb version [8].

The next step is to compute the electron propagator. To this end, having established the correspondence between the TL and NLT models, we can use the results of Section 3 in a quite direct way. Exactly as before, the non-vanishing components of the fermionic 2-point function are factorized into fermionic and bosonic contributions ( see eq (3.2) ). The only difference is that now the Fermi momentum has to be incorporated in the free-fermion factor. This is easily done and gives rise to the following change in (3.3):

$$G_{\pm}^0(z) = \frac{e^{\pm ip_F z_1} (z_0 \pm iz_1)}{2\pi |z|^2} \quad (4.10)$$

Now we must specialize eq.(3.12) for the TL model, just by inserting (4.5). This leads to

$$B_{\pm}(z) = \exp\left[\frac{1}{\pi^2} \int d^2p \frac{v(p)}{p^2} \frac{\sin^2(\frac{p \cdot z}{2})(p_0 \pm ip_1)^2}{p_0^2 + (1 + 2v/\pi)p_1^2}\right]. \quad (4.11)$$

The momentum distribution for branch 1 (2) electrons is given by

$$N_1(p_1) = \mathcal{C}(\Lambda) \int_{-\infty}^{\infty} dz_1 e^{-ip_1 z_1} \lim_{z_0 \rightarrow 0} G_{\pm}(z_0, z_1) \quad (4.12)$$

Replacing (4.10) and (4.11) in (4.12) we get

$$\begin{aligned} N_1(p_1) &= \pm \frac{\mathcal{C}(\Lambda)i}{2\pi} \int_{-\infty}^{\infty} dz_1 e^{-iz_1(p_1 \mp p_F)} \times \\ &\times \frac{1}{z_1} \exp\left\{\frac{1}{\pi^2} \int \frac{d^2p}{p^2} \frac{v(p)}{p^2} \sin^2\left(\frac{p_1 z_1}{2}\right) \frac{(p_0^2 - p_1^2)}{p_0^2 + [1 + 2v(p)/\pi]p_1^2}\right\} \end{aligned} \quad (4.13)$$

Here  $\mathcal{C}(\Lambda)$  is a normalization constant depending on an ultraviolet cutoff  $\Lambda$ . In the local limit, in which  $v(p) = \text{const}$ , the integrals in the momentum can be easily evaluated and one obtains

$$N_1(p_1) = \pm \frac{i}{2\pi} \mathcal{C}(\Lambda) \int_{-\infty}^{\infty} dz_1 \frac{e^{-i(p_1 \mp p_F)z_1}}{z_1^{1+\sigma}} \quad (4.14)$$

with

$$\sigma = \frac{1}{2} \left\{ \left(1 + \frac{2v}{\pi}\right)^{1/2} + \left(1 + \frac{2v}{\pi}\right)^{-1/2} - 2 \right\} \quad (4.15)$$

Note that in the free case  $v \rightarrow 0$  one gets  $\sigma = 0$ , which leads to the well-known normal Fermi-liquid behavior,

$$N_1 \propto \theta(p_1 \pm p_F). \quad (4.16)$$

As soon as the interaction is switched on, one has  $\sigma \neq 0$  and the Fermi edge singularity is washed out, giving rise to the so called Luttinger-liquid behavior [8][18][19]. It has been emphasized recently [26] that the experimental data obtained for one-dimensional structures can be successfully explained on the basis of standard Fermi-liquid theory. We believe that our approach could be useful to explore some modifications of the TL model to take into account, for instance, the presence of impurities or defects, that might yield a restoration of the edge singularity.

## 5 The non-Abelian case

In this section we shall show how the path-integral approach developed above can be naturally extended to the non-Abelian case. As we shall see, our results will allow us to make contact with the TL model with spin- $\frac{1}{2}$  fermions [27][28].

We shall study a non-local version of the chiral invariant Gross-Neveu model [3]. As it is well-known, in the local case a Fierz type transformation can be used to write the interaction Lagrangian in terms of  $U(N)$  generators,

$$L_{int} = -\frac{g^2}{2} J_\mu^a(x) J_\mu^a(x), \quad a = 0, 1, \dots, N^2 - 1, \quad (5.1)$$

where

$$J_\mu^a(x) = \bar{\Psi}(x) \gamma_\mu \lambda^a \Psi(x) \quad (5.2)$$

with

$$\begin{aligned} \lambda^0 &= I/2, \\ \lambda^j &= t^j, \end{aligned} \quad (5.3)$$

$t^j$  being the  $SU(N)$  generators normalized according to

$$tr (t^i t^j) = \frac{1}{2} \delta^{ij}. \quad (5.4)$$

Let us consider the action term binding fermionic currents in the form

$$S_{int} = -\frac{g^2}{2} \int d^2x d^2y J_\mu^a(x) V_{(\mu)}^{ab}(x, y) J_\mu^b(y), \quad (5.5)$$

where  $V_{(0)}^{ab}(x, y)$  and  $V_{(1)}^{ab}(x, y)$  are  $N^2 \times N^2$  matrices that weights the interaction. Exactly as we did in the Abelian case (see section 2), we can now define the set of new currents given by

$$K_\mu^a(x) = \int d^2y V_{(\mu)}^{ab}(x, y) J_\mu^b(y). \quad (5.6)$$

Introducing auxiliary fields  $\tilde{A}_\mu^a$  and  $\tilde{B}_\mu^a$  in the functional integral we arrive at

$$Z = \int D\bar{\Psi} D\Psi D\tilde{A}_\mu^a D\tilde{B}_\mu^a e^{-S_0} e^{-\int d^2x d^2y [\tilde{A}_\mu^a \tilde{B}_\mu^a - J_\mu^a \tilde{A}_\mu^a - K_\mu^a \tilde{B}_\mu^a]} \quad (5.7)$$

It is convenient to generalize definition (2.10) in the form

$$\bar{B}_\mu^b(x) = \int d^2x \tilde{B}_\mu^a V_{(\mu)}^{ab}(y, x) \quad (5.8)$$

and perform the change of variables

$$\frac{1}{\sqrt{2}}(\tilde{A}_\mu^a + \bar{B}_\mu^a) = A_\mu^a, \quad (5.9)$$

$$\frac{1}{\sqrt{2}}(\tilde{A}_\mu^b - \bar{B}_\mu^a) = B_\mu^b. \quad (5.10)$$

At this point one can easily show that  $B_\mu$  is decoupled from the fermion fields and its contribution to the partition function can be absorbed in the normalization constant, exactly as in the Abelian case. We then get

$$Z = N_2 \int DA_\mu \det(i\not{D} + g\not{A}) e^{-\frac{1}{2} \int d^2x d^2y A_\mu^a(x) A_\mu^b(y) (V_\mu^{-1})^{ab}(x, y)} \quad (5.11)$$



Now the fermionic determinant in the above expression can be treated according to the lines of ref [12]. Indeed, it is by now well-known that the decoupling change in the fermionic path-integral variables given by eqs.(2.18)-(2.21) can be readily extended to the non-Abelian case. The corresponding Jacobian gives rise to a Wess-Zumino-Witten (WZW) action term of the form

$$\log J_F = -W[hl^{-1}] - \alpha \operatorname{tr} \int d^2x \, h^{-1} \partial_+ h \, l^{-1} \partial_- l \quad (5.12)$$

where

$$W[h] = \frac{1}{8\pi} \operatorname{tr} \int d^2x \, \partial_\mu h^{-1} \partial^\mu h + \frac{1}{12\pi} \operatorname{tr} \int_B d^3y \, \epsilon_{ijk} h^{-1} \partial^i h \, h^{-1} \partial^j h \, h^{-1} \partial^k h, \quad (5.13)$$

and the second term in (5.12) is related to the regularization ambiguities which appear when one computes  $J_F$  (it is the non-Abelian analog of  $S_{reg}$  defined in (2.24)). The light-cone components of  $A_\mu$  have been written in terms of  $h$  and  $l$  (elements of  $U(N)$ ) as

$$\begin{aligned} A_+ &= h^{-1} \partial_+ h \\ A_- &= l^{-1} \partial_- l \end{aligned} \quad (5.14)$$

where  $A_\pm = A_0 \pm iA_1$ ,  $\partial_\pm = \partial_0 \pm i\partial_1$ . Of course, in order to write the partition function one also needs to consider the bosonic Jacobian  $J_A[h, l]$ , associated to (5.12) [29][30]:

$$DA_\mu \equiv DA_+ DA_- = J_A[h, l] Dh \, Dl. \quad (5.15)$$

A thorough evaluation of  $J_A$  in the light-cone gauge ( $A_- = 0$ ) has been given in ref [30]. It is quite easy to perform the calculation for the case in which no gauge choice is made. The result is

$$\log J_A = -2C\{W[hl^{-1}] + \alpha \operatorname{tr} \int d^2x \, h^{-1} \partial_+ h l^{-1} \partial_- l\}, \quad (5.16)$$

where  $C$  is the quadratic Casimir of the group under consideration,  $f^{acd} f^{bcd} = \delta^{ab} C$ . The next step is to express the non-local piece of the action (the

exponent in (5.11)) in terms of  $h$  and  $l$ , just by inserting (5.12). Replacing now (5.12) and (5.16) in (5.11), we finally get

$$Z = \mathcal{N} \int Dh \, Dl \, e^{-S_{eff}[h,l]} \quad (5.17)$$

where

$$\begin{aligned} S_{eff}[h,l] = & (2C+1)\{W[hl^{-1}] + \alpha \, tr \int d^2x \, h^{-1} \partial_+ h \, l^{-1} \partial_- l\} + \\ & + \frac{1}{2} \int d^2x \, d^2y \, \{(h^{-1} \partial_+ h)^a(x) R^{ab}(x,y) (h^{-1} \partial_+ h)^b(y) + \\ & + (l^{-1} \partial_- l)^a(x) R^{ab}(x,y) (l^{-1} \partial_- l)^b(y) + \\ & + 2(h^{-1} \partial_+ h)^a(x) T^{ab}(x,y) (l^{-1} \partial_- l)^b(y)\} \end{aligned} \quad (5.18)$$

and  $\mathcal{N}$  is the final normalization factor containing the free determinants which are left in the decoupling process. The new matrices  $R$  and  $T$  are related to  $V_{(0)}$  and  $V_{(1)}$  through

$$\begin{aligned} R &= \frac{b_{(0)} - b_{(1)}}{4} \\ T &= \frac{b_{(0)} + b_{(1)}}{4} \end{aligned} \quad (5.19)$$

where, for later convenience, we have defined  $b_{(\mu)} = V_{(\mu)}^{-1}$ .

Thus we have obtained a completely bosonized action for the non-local Gross-Neveu model with interaction given by (5.5). The analysis of the physical content of (5.18) is not trivial. We think that the background field method [31] will be useful in order to describe, at least in an approximate way, the spectrum of the bosonic excitations. We shall return to this aspect at the end of the Section.

In the present context it is very interesting to consider a simplified version of the interaction (5.5), defined by restricting the generators to those generating the maximal Abelian subgroup of  $U(N)$ . Although our procedure can be easily used for arbitrary  $N$ , here we shall pay special attention to the case  $N = 2$ . The reason for doing this is the following. For  $N = 2$  one can readily extend the arguments given in Section 4 in order to show that the

present model describes a many-body system of spin- $\frac{1}{2}$  fermions when spin flipping processes are not allowed. Indeed, choosing the potential matrices  $V_{(0)}$  and  $V_{(1)}$  as diagonal

$$\begin{aligned} V_{(0)} &= \text{diag}(v_0, v_1) \\ V_{(1)} &= \text{diag}(u_0, u_1), \end{aligned} \quad (5.20)$$

where  $v_0$ ,  $v_1$ ,  $u_0$  and  $u_1$  are written in terms of the  $g$ -functions defined by Solyom [27] as

$$\begin{aligned} v_0 &= \frac{1}{4}(g_{4\parallel} + g_{4\perp} + g_{2\parallel} + g_{2\perp}), \\ v_1 &= \frac{1}{4}(g_{4\parallel} - g_{4\perp} + g_{2\parallel} - g_{2\perp}), \\ u_0 &= \frac{1}{4}(-g_{4\parallel} - g_{4\perp} + g_{2\parallel} + g_{2\perp}), \\ u_1 &= \frac{1}{4}(-g_{4\parallel} + g_{4\perp} + g_{2\parallel} - g_{2\perp}), \end{aligned} \quad (5.21)$$

one can easily verify that the interaction term (5.5) contains the whole set of diagrams associated to forward scattering processes without spin-flips. Let us recall that the coupling constants for incident fermions with parallel spins are denoted by the subscript  $\parallel$  and that for fermions with opposite spins by the subscript  $\perp$ . In the  $g_2$  processes the two branches (left and right moving particles) are coupled, while in the  $g_4$  processes all four participating fermions belong to the same branch.

We can now specialize expression (5.11) to obtain the partition function corresponding to our restricted model. As we shall see, our task is greatly simplified in this case. In fact, as shown in Ref [12], the fermionic determinant is decoupled just by extending the Abelian change of variables in the following simple way

$$\begin{aligned} \Psi &= e^{-g[\gamma_5 \Phi + i\eta]} \chi \\ \bar{\Psi} &= \bar{\chi} e^{-g[\gamma_5 \Phi - i\eta]} \\ A_\mu &= \epsilon_{\mu\nu} \partial_\nu \Phi + \partial_\mu \eta \end{aligned} \quad (5.22)$$

with  $\Phi = \Phi^a \lambda^a, \eta = \eta^a \lambda^a, a = 0, 1$ . ( $\lambda^0$  and  $\lambda^1$  are the generators of the maximal Abelian subgroup). In this case the bosonic Jacobian is trivial, while the fermionic one is given by

$$\log J_F = \frac{g^2 + \alpha}{2\pi} \int d^2x \Phi^a \square \Phi^a, \quad (5.23)$$

where  $\alpha$  is a regularization parameter. Note that, having restricted the model to the Cartan subalgebra, no WZW term appears in  $J_F$ , as expected.

One can now write the effective bosonized action of the model which, in terms of the Fourier transforms of  $\Phi^a, \eta^a$  and  $b_{(\mu)}^{ab}$  ( $\hat{\Phi}^a, \hat{\eta}^a$  and  $\hat{b}_{(\mu)}^{ab}$ ) reads

$$\begin{aligned} S_{eff} = \frac{1}{2} \int \frac{d^2p}{(2\pi)^2} \{ & \hat{\Phi}^a(p) \hat{\Phi}^b(-p) [p_1^2 \hat{b}_{(0)}^{ab}(p) + p_0^2 \hat{b}_{(1)}^{ab}(p) + 2\lambda \delta^{ab} p^2] \\ & + \hat{\eta}^a(p) \hat{\eta}^b(-p) [p_1^2 \hat{b}_{(1)}^{ab}(p) + p_0^2 \hat{b}_{(0)}^{ab}(p)] \\ & - 2\hat{\Phi}^a(p) \hat{\eta}^b(-p) p_0 p_1 [\hat{b}_{(1)}^{ab}(p) - \hat{b}_{(0)}^{ab}(p)] \}. \end{aligned} \quad (5.24)$$

where  $\lambda$  is defined exactly as in the Abelian case (see eq.(2.34)). At this point we observe that the above action can be expressed in terms of a vector field  $(\hat{\Phi}^0, \hat{\eta}^0, \hat{\Phi}^1, \hat{\eta}^1)$ . In doing so we obtain an interaction matrix with a block diagonal form: each block is a mere reproduction of that corresponding to the Abelian case (see Section 2). Consequently we get a couple of dispersion relations for the collective modes of the system:

$$\begin{aligned} \omega_\rho^2 &= k^2 \frac{1 + 2\lambda v_0}{1 + 2\lambda u_0} \\ \omega_\sigma^2 &= k^2 \frac{1 + 2\lambda v_1}{1 + 2\lambda u_1} \end{aligned} \quad (5.25)$$

where  $p_1 = k$  and  $\omega = ip_0$ .

The first one describes the charge-density fluctuations  $(\hat{\Phi}^0, \hat{\eta}^0)$ , whereas the second one is associated to spin-density modes  $(\hat{\Phi}^1, \hat{\eta}^1)$ . This identification can be understood by recalling that charge and spin densities for branch  $i$  ( $i = 1, 2$ ) particles are defined as

$$\begin{aligned}
\rho_i &= \Psi_{i\uparrow}^\dagger \Psi_{i\uparrow} + \Psi_{i\downarrow}^\dagger \Psi_{i\downarrow} \\
\sigma_i &= \Psi_{i\uparrow}^\dagger \Psi_{i\uparrow} - \Psi_{i\downarrow}^\dagger \Psi_{i\downarrow}
\end{aligned} \tag{5.26}$$

and the fermionic currents  $J_\mu^a$  ( $a = 0, 1$ ) are connected to the above densities through

$$\begin{aligned}
J_0^0 &= \rho_1 + \rho_2 \\
J_1^0 &= i(\rho_2 - \rho_1) \\
J_0^1 &= \sigma_1 + \sigma_2 \\
J_1^1 &= i(\sigma_2 - \sigma_1)
\end{aligned} \tag{5.27}$$

These densities, in turn, can be written in terms of  $\phi$  and  $\eta$  fields by using the usual bosonization identity  $J_\mu \propto A_\mu$ .

Concerning the dispersion relations (5.25), it is interesting to consider the limit  $u_0 \rightarrow 0$ ,  $u_1 \rightarrow 0$  which corresponds to an interaction including density-density fluctuations only. In this case our result gives the spectrum of the spin- $\frac{1}{2}$  TL model [27],[25].

After the precedent example, it becomes clear that the complete interaction (5.5) (without restricting the generators to the Cartan subalgebra) can be used to study the TL model when spin-flipping processes are taken into account [28]. In fact, setting  $N = 2$  in (5.18), we obtain the bosonized effective action for this system. However, due to the presence of the Wess-Zumino term, the analysis of the spectrum associated to density oscillations is not a trivial task in this case. One possible strategy to face this problem is to consider quantum fluctuations around a classical background field [31]. This work is beyond the scope of the present article, but will be addressed in the close future [32].

## 6 Conclusions

In this paper we have shown how to extend the path-integral approach to bosonization, previously developed in the context of 2-d local QFT's [11], [12], to the case in which the fermionic interaction is non-local. In particular we have considered the Thirring-like model defined by equation (1.1). By introducing two auxiliary vector fields we were able to write the corresponding partition function in terms of a fermionic determinant. This fact allowed us to employ the well established procedure of functional bosonization, based on a decoupling change of path-integral variables, in order to get a completely bosonized action (eq. (2.27)). We have also computed the 2-point fermionic correlation function (formulae (3.2) and (3.12)). We want to stress that our results are valid for arbitrary bilocals  $V_{(\mu)}(x, y)$ , i.e. we do not need to specify the potentials in order to get closed formulae for the bosonized action and Green's functions.

In Section 4 we have applied our approach to the finite-temperature formulation of the Tomonaga-Luttinger model which describes a system of 1-d spinless fermions with a linearized dispersion relation. We have shown that, for one particular choice of the bilocal potentials  $V_{(\mu)}(x, y)$ , the NLT model contains the same forward scattering processes that are present in the zero temperature limit of the TL system. Therefore, we used the procedure developed in Sections 2 and 3 to obtain the bosonized action and 2-point fermionic correlators corresponding to the TL model. In this context, our effective bosonic action correctly describes the plasma oscillations associated to charge-density fluctuations. Concerning the fermionic Green's function, we have also obtained the characteristic TL behavior leading to the disappearance of the edge singularity in the momentum distribution.

Finally, in Section 5, we have performed the extension of the path-integral approach to the bosonization of non-local interactions involving non-Abelian groups. In order to illustrate the procedure, we have considered a simple generalization of the Abelian action (1.1), which is obtained by assigning a group index to the currents  $J_\mu^a$  and promoting the potentials  $V_{(\mu)}(x, y)$  to a matrix potential  $V_{(\mu)}^{ab}(x, y)$  ( $a, b = 0, \dots, N_G$ , where  $N_G$  is the number of generators of the group under consideration). Exactly as in the Abelian case the decoupling change in the path-integral variables can be successfully done.

Thus we found a bosonized action containing a WZW functional (eq.(5.5)). As it was to be expected, the analysis of the physical spectrum is not as straightforward in this case as it was in the Abelian model. We have then considered a simpler but still interesting version of the model. First of all we have observed that, setting the group as  $U(2)$ , the non-Abelian action (5.5) contains the same forward scattering processes as the spin-1/2 TL model, without spin-flip. We have considered a simplified model defined by restricting the generators to those generating the maximal Abelian subgroup of  $U(2)$  (in the TL language this means that fermions do not change their spins in the scattering process). When only density-density fluctuations are taken into account our method allowed us to get a bosonic action describing decoupled charge and spin density fluctuations (eqs.(5.24) and (5.25)).

In summary, we have presented a bosonization procedure that can be used even when arbitrary non-local interactions are included in the original 2-dimensional fermionic action. We have shown how to take advantage of this method in order to analyze 1-d many-body systems in an elegant way. Concerning the TL model, our results agree with previous calculations performed in the operational framework [8] and with different functional techniques [25]. We feel, however, that our approach will be specially useful to study spin-flipping interactions. Backward scattering and umklapp diagrams could be also included in this formulation. We hope to report on these issues in a forthcoming publication.

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